Vector Geometry

Fact —

- $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ ("end minus start", with position vectors from O).
- For a column vector, $\left|\begin{pmatrix} p \\ q \end{pmatrix}\right| = \sqrt{p^2 + q^2}$.
- Two vectors are **parallel** exactly when one is a scalar multiple of the other.
- A, B, C are **collinear** when $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ (they already share B).
- Midpoint of AB: position vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Example

$\mathbf{u} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 6 \\ k \end{pmatrix}$.

1. Find $|\mathbf{u}|$.
2. Find k such that $\mathbf{v}$ is parallel to $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$.
3. Find both values of k such that $|\mathbf{v}| = 10$.

1. $\sqrt{25 + 144} = 13$ *2.* $\frac{6}{9} = \frac{k}{12} \implies k = 8$ *3.* $36 + k^2 = 100 \implies k = \pm 8$

Example

Relative to an origin O, the points A, B, C have position vectors

$$\mathbf{a} + 2\mathbf{b}, \qquad 3\mathbf{a} + 3\mathbf{b}, \qquad 7\mathbf{a} + 5\mathbf{b}$$

where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. Prove that A, B and C are collinear, and state the ratio $AB : BC$.

$\overrightarrow{AB} = 2\mathbf{a} + \mathbf{b} \qquad \overrightarrow{BC} = 4\mathbf{a} + 2\mathbf{b} = 2\overrightarrow{AB}$

$\overrightarrow{BC}$ is a multiple of $\overrightarrow{AB}$ and they share the point B, so A, B, C are collinear, with $AB : BC = 1 : 2$.

Example

$OACB$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of AC and N is the point on BC with $BN : NC = 3 : 1$. Express in terms of $\mathbf{a}$ and $\mathbf{b}$:

1. $\overrightarrow{OC}$
2. $\overrightarrow{OM}$
3. $\overrightarrow{MN}$

1. $\mathbf{a} + \mathbf{b}$
2. $\overrightarrow{AC} = \mathbf{b}$, *so* $\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}\mathbf{b}$
3. $\overrightarrow{ON} = \mathbf{b} + \frac{3}{4}\mathbf{a}$, *so* $\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = -\frac{1}{4}\mathbf{a} + \frac{1}{2}\mathbf{b}$

Textbook Exercises: SPS Course 2.6, Exercises 4A and 4B

Parameters

When a point lies on a line but its position is unknown, introduce a parameter: X on the line through B and M means $\overrightarrow{BX} = \mu\overrightarrow{BM}$ for some scalar μ.

Fact — If $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and

$$\lambda_1\mathbf{a} + \mu_1\mathbf{b} = \lambda_2\mathbf{a} + \mu_2\mathbf{b},$$

then $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$: coefficients can be equated.

Example

In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of OA, and N lies on AB with $AN : NB = 1 : 2$. The lines ON and BM intersect at X.
Find $\overrightarrow{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, and hence the ratio $OX : XN$.

$\overrightarrow{ON} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$
X on ON: $\overrightarrow{OX} = \lambda\left(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)$
X on BM: $\overrightarrow{OX} = \mathbf{b} + \mu\left(\frac{1}{2}\mathbf{a} - \mathbf{b}\right) = \frac{\mu}{2}\mathbf{a} + (1-\mu)\mathbf{b}$
Equate coefficients: $\frac{2\lambda}{3} = \frac{\mu}{2}$ *and* $\frac{\lambda}{3} = 1 - \mu$.
Substituting $\mu = \frac{4\lambda}{3}$: $\frac{\lambda}{3} = 1 - \frac{4\lambda}{3} \implies \lambda = \frac{3}{5}$.
$\overrightarrow{OX} = \frac{2}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}$, *and* $\lambda = \frac{3}{5}$ *gives* $OX : XN = 3 : 2$.

Example

$OACB$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. D is the midpoint of AC. The line OD is extended to meet the line BC extended at E.
Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, and show that C is the midpoint of BE.

$\overrightarrow{OD} = \mathbf{a} + \frac{1}{2}\mathbf{b}$.
E on OD extended: $\overrightarrow{OE} = s\left(\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$.
E on line BC: $\overrightarrow{OE} = \mathbf{b} + u\,\mathbf{a}$ *(since* $\overrightarrow{BC} = \mathbf{a}$*).*
Equate coefficients: $\frac{s}{2} = 1 \implies s = 2$, *and* $u = s = 2$.
$\overrightarrow{OE} = 2\mathbf{a} + \mathbf{b}$. $\overrightarrow{BE} = 2\mathbf{a} = 2\overrightarrow{BC}$, *so C is the midpoint of BE.*

Textbook Exercises: SPS Course 2.6, Exercises 4C and 4D